Solutions to Midterm

1. a) $P(A \cup B \cup C) = 1 - P(\text{none occurs})$

$$= 1 - P(A^c \cap B^c \cap C^c)$$

$$= 1 - P(A^c)\,P(B^c)\,P(C^c)$$

$$= 1 - (0.5)(0.8)(0.7) = 0.72$$

b) $P(E^c \cap F^c) = P((E \cup F)^c) = (1 - 0.75)) = 0.25$ since

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.75$$

2. For $n = 2$, $P(E_1 \cap E_2) = P(E_1) + P(E_2) - P(E_1 \cup E_2)$

$$= \left[1 - P(E_1 \cup E_2)\right] + P(E_1) + P(E_2) - 1$$

$$\geq P(E_1) + P(E_2) - 1$$

Assume for n and we now show it for $n+1$

$$P\left(\bigcap_1^{n+1} E_i\right) = P\left(\bigcap^{n} E_i \cap E_{n+1}\right) = P\left(\bigcap^{n} E_i\right) + P(E_{n+1}) - P\left(\bigcap E_i \cup E_{n+1}\right)$$

$$\geq \sum^{n+1} P(E_i) - P\left(\bigcap E_i \cup E_{n+1}\right) - (n-1)$$

induction hypo

$$\geq \sum_1^{n+1} P(E_i) + 1 - P\left(\bigcap^{n} E_i \cup E_{n+1}\right) - (n+1-1)$$

$$\geq \sum^{n+1} P(E_i) - (n+1-1)$$

3. A = 1st ace is the 10th card drawn

B = 11th card is ace of spades

$\underset{1}{\Box}\ \underset{2}{\Box}\ \cdots\ \underset{10}{\Box}\ \underset{11}{\Box}$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = \frac{(3)(1)(48)_9}{(52)_{21}}, \quad P(A) = \frac{(4)(42)\cdot(48)_9}{(52)_{21}}$$

$$\therefore\ P(B \mid A) = \frac{3}{4(42)} = \frac{3}{168}$$

Notation $(n)_r = n(n-1)\cdots(n-r+1)$

4. In the gambler's ruin problem, let E be the probability that you will win all the money. Then we showed

$$P(E) = P_i = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 1/2 \\[2ex] i/N & \text{if } p = 1/2 \end{cases}$$

a) If $p = 1/2$, $P_i \to 0$ as $N \to \infty$

If $p < 1/2$, $(q/p) > 1$ and hence $(q/p)^N \to \infty$ as $N \to \infty$

and so $P_i \to 0$

b) If $p > 1/2$, $(q/p) < 1$ and hence $(q/p)^N \to 0$ as $N \to \infty$

and so $P_i \to 1 - (q/p)^i$

5. Let A be the event that the first is black and
B " " " " second " " .

a) $$P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{(2)(1)/(5)(4)}{(2)(4)/(5)(4)} = 1/4$$

b) $$P(B^c \cap A^c) + P(B \cap A) = \frac{(3)(2)}{(5)(4)} + \frac{(2)(1)}{(5)(4)} = \frac{8}{20}$$

c) $$P(A^c \mid B^c) = \frac{P(A^c \cap B^c)}{P(B^c)} = \frac{(3)(2)/(5)(4)}{(4)(3)/(5)(4)} = \frac{6}{12} = \frac{1}{2}$$